

FUNCTORIALITY OF SHEAF CATEGORIES OVER WEINSTEIN MANIFOLDS

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ABSTRACT. We consider the functoriality of microlocal sheaf category over Weinstein sectors defined by Nadler-Shende. In particular, we strengthen Nadler-Shende's invariance result and show that the microlocal sheaf category is invariant under all Liouville homotopies.

1. RESULTS

In the recent work [9], Nadler-Shende defined the microlocal sheaf category associated to Weinstein sectors with Maslov data, where exact Lagrangian submanifolds arise as objects in the category, without using arborealization. However, they have technical difficulties in proving functoriality of the embedding functor (i.e. the gapped microlocal specialization functor). Here we resolve this issue by slightly adapting the proof in our previous work [6, Theorem 3.2].

Definition 1.1. *Let X_0 and X_1 be Liouville sectors. Then a Liouville subsector embedding is an embedding sending sectorial boundary to sectorial boundary, i.e. $\partial X_0 \subset \partial X_1$, such that $X_1 \setminus X_0$ is a sutured or sectorial Liouville cobordism from $\partial_\infty X_0$ to $\partial_\infty X_1$.*

We remark that our definition of Liouville subsector embeddings is different from embeddings of Liouville sectors in literature, which send contact boundaries to contact boundaries [1]. For Liouville domains however, this agrees with the standard notion of Liouville embeddings of domains. See also [5, Section 2.6 & 2.12] for the discussion.

Theorem 1.1. *Let X_0, X_1, X_2 be Weinstein sectors with Lagrangian skeleta $\mathbf{c}_{X_0}, \mathbf{c}_{X_1}, \mathbf{c}_{X_2}$ equipped with Maslov data, such that $i_{01} : X_0 \hookrightarrow X_1$ and $i_{12} : X_1 \hookrightarrow X_2$ are Liouville embeddings sending sectorial boundaries to sectorial boundaries. Denote by $\Phi_{ij} : \mu Sh_{\mathbf{c}_{X_i}}(\mathbf{c}_{X_i}) \hookrightarrow \mu Sh_{\mathbf{c}_{X_j}}(\mathbf{c}_{X_j})$ the embedding of microlocal sheaf categories. Then*

$$\Phi_{12} \circ \Phi_{01} \simeq \Phi_{02} : \mu Sh_{\mathbf{c}_{X_0}}(\mathbf{c}_{X_0}) \hookrightarrow \mu Sh_{\mathbf{c}_{X_2}}(\mathbf{c}_{X_2}).$$

Our strategy is as follows. Φ_{02} is defined by using the Liouville flow to compress \mathbf{c}_{X_0} to the ambient skeleton \mathbf{c}_{X_2} directly, and $\Phi_{12} \circ \Phi_{01}$ is defined by first compressing \mathbf{c}_{X_0} to the skeleton \mathbf{c}_{X_1} , and next compressing \mathbf{c}_{X_1} to the ambient skeleton \mathbf{c}_{X_2} . We will try to define a 2-parametric family of contact flow that interpolates between them. Then following the construction, Φ_{01} and $\Phi_{12} \circ \Phi_{01}$ are two different compositions of nearby cycles, and the theorem is reduced to commutativity of the nearby cycle functors.

Therefore, we need the commutativity criterion of nearby cycle functors in for example [8] or [4, 7]. In order to keep the proof self contained, we extract the main technical lemma as follows, which is a base change formula that does not hold in general. Write the projection maps

$$\pi_i : N \times [0, 1] \times [0, 1] \rightarrow [0, 1], \quad (x, t_1, t_2) \mapsto t_i, \quad (i = 1, 2)$$

and $\pi = \pi_1 \times \pi_2 : N \times [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$. Write the inclusions

$$\begin{array}{ccc} N \times \{0\} \times (0, 1] & \xrightarrow{i} & N \times [0, 1] \times (0, 1] \\ j \downarrow & & \downarrow \bar{j} \\ N \times \{0\} \times [0, 1] & \xrightarrow{\bar{j}} & N \times [0, 1] \times [0, 1]. \end{array}$$

Recall that for a closed embedding $i : N \hookrightarrow M$ and a subset $A \subset T^*M$, we define $i^\#(A) \subset T^*N$ to be the points $(x, \xi) \in T^*N$ such that there exists $(y_n, \eta_n, x_n, 0) \in T^*N \times T^*M$ and

$$x_n, y_n \rightarrow x, i^* \eta_n \rightarrow \xi, |x_n - y_n| |\eta_n| \rightarrow 0.$$

Proposition 1.2. *Let $\mathcal{F} \in Sh(N \times [0, 1] \times (0, 1])$ be a sheaf such that*

- (1) $i^\# SS^\infty(\mathcal{F}) \cap \pi_2^* T^{*,\infty}(0, 1] = \emptyset$,
- (2) $SS^\infty(\mathcal{F}) \cap \pi^* T^{*,\infty}((0, 1] \times (0, 1]) = \emptyset$,
- (3) $\overline{SS_\pi^\infty(\mathcal{F})} \cap T^{*,\infty} N \times \{(0, 0)\}$ is a subanalytic Legendrian.

Then there is a natural isomorphism of sheaves

$$\bar{i}^{-1} \bar{j}_* \mathcal{F} \simeq j_* i^{-1} \mathcal{F}.$$

Remark 1.1. *For the applications, \mathcal{F} will always be the push forward of a sheaf $\mathcal{F}_0 \in Sh(N \times (0, 1] \times (0, 1])$, in which case Condition (1) can be easily checked. We choose to state a more general result without assuming that.*

Remark 1.2. *We remark the importance of Condition (3). The following example is due to an anonymous referee. Let $N = \mathbb{R}$, $S = \{(x, t_1, t_2) | t_1 = xt_2\} \subset N \times [0, 1] \times (0, 1]$ and $\mathcal{F} = \mathbb{k}_S$. Then Condition (3) does not hold and one can check that the base change formula does not hold.*

We have a natural morphism $\bar{i}^{-1} \bar{j}_* \mathcal{F} \rightarrow j_* i^{-1} \mathcal{F}$ by adjunction. Since the natural morphism induces quasi-isomorphisms on stalks on $N \times 0 \times (0, 1]$, it suffices to show that the it also induces quasi-isomorphisms on stalks on $N \times \{(0, 0)\}$.

First we compute the stalks of $\bar{i}^{-1} \bar{j}_* \mathcal{F}$ at $(x, 0, 0)$. The following lemma is basically [9, Corollary 4.4]. Let U_x be a sufficiently small open ball around $x \in N$, $D_{(0,0)}(\epsilon) = [0, \epsilon) \times [0, \epsilon)$, $D_{(0,0)}^\circ(\epsilon) = [0, \epsilon) \times (\delta, \epsilon)$, and respectively \bar{U}_x , $\bar{D}_{(0,0)}(\epsilon)$ and $\bar{D}_{(0,0)}^\circ(\epsilon)$ be their closures.

Lemma 1.3. *Let $\mathcal{F} \in Sh(N \times [0, 1] \times (0, 1])$ be a sheaf so that $i^\# SS^\infty(\mathcal{F}) \cap \pi_2^* T^{*,\infty}(0, 1] = \emptyset$, $SS^\infty(\mathcal{F}) \cap \pi^* T^{*,\infty}((0, 1] \times (0, 1]) = \emptyset$, and $\overline{SS_\pi^\infty(\mathcal{F})} \cap T^{*,\infty} N \times \{(0, 0)\}$ is a subanalytic Legendrian. Then for $x \in N$, $U_x \subset N$ a sufficiently small open neighbourhood and $\epsilon > 0$ sufficiently small,*

$$\bar{j}_* \mathcal{F}_{(x,0,0)} \simeq \Gamma(\bar{U}_x \times \bar{D}_{(0,0)}^\circ(\epsilon), \mathcal{F}).$$

Proof. Since $\overline{SS_\pi^\infty(\mathcal{F})} \cap T^{*,\infty} N \times \{(0, 0)\}$ is a subanalytic Legendrian, for any sufficiently small neighbourhood U_x of $x \in N$, we have

$$\overline{SS_\pi^\infty(\mathcal{F})} \cap \nu_{U_x, \pm}^{*,\infty} N \times \{(0, 0)\} = \emptyset$$

by general position argument.

Consider $N \times (0, 1] \times [0, 1]$. Since $SS^\infty(\mathcal{F}) \cap \pi^* T^{*,\infty}((0, 1] \times (0, 1]) = \emptyset$, we can get an injective projection to the relative singular support in the relative cotangent bundle $SS^\infty(\mathcal{F}) \hookrightarrow SS_\pi^\infty(\mathcal{F})$ on $N \times (0, 1] \times (0, 1]$. Hence there is an injective projection

$$SS^\infty(\mathcal{F}) \cap \nu_{U_x \times D_{(0,0)}(\epsilon), \pm}^{*,\infty}(N \times (0, 1] \times (0, 1]) \hookrightarrow SS_\pi^\infty(\mathcal{F}) \cap \nu_{U_x, \pm}^{*,\infty} N \times D_{(0,0)}(\epsilon).$$

Then consider $N \times \{0\} \times (0, 1]$. Since $i^\# SS^\infty(\mathcal{F}) \cap \pi_2^* T^{*,\infty}(0, 1] = \emptyset$ and $\nu_{U_x \times \{0\} \times D_0(\epsilon), \pm}^{*,\infty}(N \times \{0\} \times (0, 1])$ only consists of covectors tangent to $N \times \{0\} \times (0, 1]$, there is also an injection $SS^\infty(\mathcal{F}) \cap \nu_{U_x \times \{0\} \times D_0(\epsilon), \pm}^{*,\infty}(N \times \{0\} \times (0, 1]) \hookrightarrow i^\# SS^\infty(\mathcal{F}) \cap \nu_{U_x \times \{0\} \times D_0(\epsilon), \pm}^{*,\infty}(N \times \{0\} \times (0, 1])$

where $D_0(\epsilon) = [0, \epsilon)$. Then by the assumption $i^\# SS^\infty(\mathcal{F}) \cap \pi_2^* T^{*,\infty}(0, 1] = \emptyset$, we have an injective projection

$$i^\# SS^\infty(\mathcal{F}) \cap \nu_{U_x \times \{0\} \times D_0(\epsilon), \pm}^{*,\infty}(N \times \{0\} \times (0, 1]) \hookrightarrow i^\# SS_{\pi_2}^\infty(\mathcal{F}) \cap \nu_{U_x, \pm}^{*,\infty} N \times \{0\} \times D_0(\epsilon).$$

Combining the two cases of $N \times (0, 1] \times [0, 1]$ and $N \times \{0\} \times (0, 1]$, we obtain an injective projection

$$SS^\infty(\mathcal{F}) \cap \nu_{U_x \times D_{(0,0)}(\epsilon), \pm}^{*,\infty}(N \times [0, 1] \times (0, 1]) \hookrightarrow SS_\pi^\infty(\mathcal{F}) \cap \nu_{U_x, \pm}^{*,\infty} N \times D_{(0,0)}(\epsilon).$$

However, as $\epsilon \rightarrow 0$ the limit points in the above relative singular support are contained in $\overline{SS_\pi^\infty(\mathcal{F})} \cap \nu_{U_x, \pm}^{*,\infty} N \times \{(0, 0)\} = \emptyset$. Therefore, the set of the limit points in the intersection of the relative singular support and the conormal bundle is empty. Hence we can conclude that for sufficiently small $\epsilon > 0$,

$$SS^\infty(\mathcal{F}) \cap \nu_{U_x \times D_{(0,0)}(\epsilon), \pm}^{*,\infty}(N \times [0, 1] \times (0, 1]) = \emptyset.$$

Consequently, by non-characteristic deformation lemma applied to the family $U_x \times D_{(0,0)}(\epsilon)$ and $U_x \times D_{(0,0)}^\circ(\epsilon)$ for sufficiently small $\epsilon > 0$ and $\delta \ll \epsilon$, we can conclude that

$$\begin{aligned} \bar{j}_* \mathcal{F}_{(x,0,0)} &\simeq \Gamma(U_x \times D_{(0,0)}(\epsilon), \bar{j}_* \mathcal{F}) \simeq \Gamma(\bar{U}_x \times \bar{D}_{(0,0)}(\epsilon), \bar{j}_* \mathcal{F}) \\ &\simeq \Gamma(U_x \times D_{(0,0)}^\circ(\epsilon), \mathcal{F}) \simeq \Gamma(\bar{U}_x \times \bar{D}_{(0,0)}^\circ(\epsilon), \mathcal{F}). \end{aligned} \quad \square$$

Then we compute the stalks of $j_* i^{-1} \mathcal{F}$ at $(x, 0)$. Let U_x be a sufficiently small open ball around $x \in N$, $D_0 = [0, \epsilon)$, $D_0^\circ = (\delta, \epsilon)$, and respectively \bar{U}_x , $\bar{D}_0(\epsilon)$, $\bar{D}_0^\circ(\epsilon)$ be their closures.

Lemma 1.4. *Let $\mathcal{G} \in Sh(N \times (0, 1])$ be a sheaf such that $SS^\infty(\mathcal{G}) \cap \pi^* T^{*,\infty}(0, 1] = \emptyset$, and $SS_\pi^\infty(\mathcal{G}) \cap T^{*,\infty} N \times \{0\}$ is subanalytic Legendrian. Then for any $x \in N$, $U_x \subset N$ a sufficiently small open neighbourhood and $\epsilon > 0$ sufficiently small,*

$$j_* \mathcal{G}_{(x,0)} \simeq \Gamma(\bar{U}_x \times \bar{D}_0^\circ(\epsilon), \mathcal{G}).$$

Proof. Since $\overline{SS_\pi^\infty(\mathcal{G})} \cap T^{*,\infty} N \times \{0\}$ is a subanalytic Legendrian, for a small neighbourhood U_x of $x \in N$, we have

$$\overline{SS_\pi^\infty(\mathcal{G})} \cap \nu_{U_x, \pm}^{*,\infty} N \times \{0\} = \emptyset$$

by general position argument. Since $SS^\infty(\mathcal{G}) \cap \pi^* T^{*,\infty}(0, 1] = \emptyset$, we have an injective projection to the relative singular support in the relative cotangent bundle $SS^\infty(\mathcal{G}) \hookrightarrow SS_\pi^\infty(\mathcal{G})$. Hence there is an injective projection

$$SS^\infty(\mathcal{G}) \cap \nu_{U_x \times D_0(\epsilon), \pm}^{*,\infty}(N \times (0, 1]) \hookrightarrow SS_\pi^\infty(\mathcal{G}) \cap \nu_{U_x, \pm}^{*,\infty} N \times D_0(\epsilon).$$

as $\epsilon \rightarrow 0$ the limit points in the above relative singular support are contained in $\overline{SS_\pi^\infty(\mathcal{G})} \cap \nu_{U_x, \pm}^{*,\infty} N \times 0 = \emptyset$. Hence we can conclude that when $\epsilon > 0$ is sufficiently small, the intersection between relative singular support and $\nu_{U_x, \pm}^{*,\infty} N \times D_0(\epsilon)$ is empty. Therefore, by non-characteristic deformation lemma applied to the family $U_x \times D_0(\epsilon)$ and $U_x \times D_0^\circ(\epsilon)$, we have

$$\begin{aligned} j_* \mathcal{G}_{(x,0)} &\simeq \Gamma(U_x \times D_0(\epsilon), j_* \mathcal{G}) \simeq \Gamma(\bar{U}_x \times \bar{D}_0(\epsilon), j_* \mathcal{G}) \\ &\simeq \Gamma(U_x \times D_0^\circ(\epsilon), \mathcal{G}) \simeq \Gamma(\bar{U}_x \times \bar{D}_0^\circ(\epsilon), \mathcal{G}). \end{aligned} \quad \square$$

Remark 1.3. *The above lemmas will also follow from the weak constructibility of \mathcal{F} [8, Section 2]. For the applications, we believe that in fact both conditions hold.*

Proof of Proposition 1.2. We apply Lemma 1.3 to \mathcal{F} and apply Lemma 1.4 to $i^{-1}\mathcal{F}$, and note that $SS^\infty(i^{-1}\mathcal{F}) \subset i^\#SS^\infty(\mathcal{F})$. Then it suffices to show that

$$\Gamma(\overline{U_x \times \overline{D}_{(0,0)}^\circ}(\epsilon), \mathcal{F}) \simeq \Gamma(\overline{U_x \times \overline{D}_0^\circ}(\epsilon), \mathcal{F}).$$

Since $\overline{SS_\pi^\infty(\mathcal{F})} \cap T^{*,\infty}N \times \{(0,0)\}$ is a subanalytic Legendrian, for a small neighbourhood U_x of $x \in N$, we have

$$\overline{SS_\pi^\infty(\mathcal{F})} \cap \nu_{U_x, \pm}^{*,\infty}N \times \{(0,0)\} = \emptyset$$

by general position argument. Write $D_{(0,0)}^\circ(\epsilon, \epsilon') = [0, \epsilon'] \times (\delta, \epsilon)$ for $0 \leq \epsilon' \leq \epsilon$. Since $SS^\infty(\mathcal{F}) \cap \pi^*T^{*,\infty}((0,1] \times (0,1]) = \emptyset$, we know that there is an injective projection onto the relative singular support

$$SS^\infty(\mathcal{F}) \cap \nu_{U_x \times D_{(0,0)}^\circ(\epsilon, \epsilon'), \pm}^{*,\infty}(N \times (0,1] \times (0,1]) \hookrightarrow SS_\pi^\infty(\mathcal{F}) \cap \nu_{U_x, \pm}^{*,\infty}N \times \overline{D}_{(0,0)}^\circ(\epsilon, \epsilon').$$

However, as $\epsilon, \epsilon' \rightarrow 0$, the limit points of the relative singular support are contained in $\overline{SS_\pi^\infty(\mathcal{F})} \cap \nu_{U_x, \pm}^{*,\infty}N \times \{(0,0)\} = \emptyset$. Therefore, the set of the limit points in the intersection of the relative singular support and the conormal bundle is empty. Hence we can conclude that when $\epsilon, \epsilon' > 0$ are sufficiently small,

$$SS^\infty(\mathcal{F}) \cap \nu_{U_x \times D_{(0,0)}^\circ(\epsilon, \epsilon'), \pm}^{*,\infty}(N \times (0,1] \times (0,1]) = \emptyset.$$

By non-characteristic deformation lemma applied to the family $D_{(0,0)}^\circ(\epsilon, \epsilon')$, we can conclude that

$$\Gamma(\overline{U_x \times \overline{D}_{(0,0)}^\circ}(\epsilon), \mathcal{F}) \simeq \Gamma(\overline{U_x \times \overline{D}_0^\circ}(\epsilon, \epsilon'), \mathcal{F}) \simeq \Gamma(\overline{U_x \times \overline{D}_0^\circ}(\epsilon), \mathcal{F}).$$

This completes the proof. \square

Remark 1.4. *When applying non-characteristic deformation lemma, one should notice that $\partial(\overline{U_x \times \overline{D}_{(0,0)}^\circ}(\epsilon))$ is piecewise smooth. Therefore, we need to use the condition that $SS^\infty(\mathcal{F}) \cap \pi^*T^{*,\infty}((0,1] \times (0,1]) = \emptyset$ rather than only considering the intersection with $\pi_1^*T^{*,\infty}(0,1]$ and $\pi_2^*T^{*,\infty}(0,1]$. For the same reason, we need the estimate on $\overline{SS_\pi^\infty(\mathcal{F})} \cap T^{*,\infty}N \times \{(0,0)\}$ rather than estimates on $\overline{SS_{\pi_1}^\infty(\mathcal{F})}$ and $\overline{SS_{\pi_2}^\infty(\mathcal{F})} \cap T^{*,\infty}N \times \{(0,0)\}$. The author is grateful to an anonymous referee for pointing out the mistake in the proposition.*

We can start the proof of the theorem. Let λ_i be the Liouville form, Z_i the Liouville vector field, and $\varphi_{Z_i}^z$ the Liouville flow on the Weinstein sector X_i . Consider the lifting of the flow $\varphi_{Z_i}^z$ in $T^{*,\infty}N$ that satisfies

$$d\varphi_{Z_i}^z/dz = t\partial/\partial t + Z_{\lambda_i}$$

on $X_i \times \mathbb{R}$. Then we know that

$$\lim_{z \rightarrow -\infty} \varphi_{Z_1}^z(\mathbf{c}_{X_0}) \subset \mathbf{c}_{X_1}, \quad \lim_{z \rightarrow -\infty} \varphi_{Z_2}^z(\mathbf{c}_{X_0}), \quad \lim_{z \rightarrow -\infty} \varphi_{Z_2}^z(\mathbf{c}_{X_1}) \subset \mathbf{c}_{X_2}.$$

Write $\phi_{Z_i}^\zeta = \varphi_{Z_i}^{\ln \zeta}$. Now consider the 2-parameter family of contact Hamiltonian $\phi_{\overline{Z}}^{\zeta, \eta} = \phi_{Z_2}^\zeta \circ \phi_{Z_1}^{\eta - \zeta}$. Then $\phi_{\overline{Z}}^{\zeta, \zeta} = \varphi_{Z_2}^\zeta$, $\phi_{\overline{Z}}^{1, \eta} = \varphi_{Z_1}^\eta$. In particular, the limits satisfy

$$\begin{aligned} \lim_{\zeta \rightarrow 0} \phi_{\overline{Z}}^{\zeta, \eta}(-) &= \lim_{\zeta \rightarrow 0} \phi_{Z_2}^\zeta(-) = \lim_{z \rightarrow -\infty} \varphi_{Z_2}^z(-), \\ \lim_{\eta \rightarrow 0} \phi_{\overline{Z}}^{\zeta, \eta}(-) &= \phi_{Z_2}^\zeta \left(\lim_{\eta \rightarrow 0} \phi_{Z_1}^\eta(-) \right) = \phi_{Z_2}^\zeta \left(\lim_{y \rightarrow -\infty} \varphi_{Z_1}^y(-) \right). \end{aligned}$$

Write $\Delta = \{(\zeta, \eta) | 0 < \eta \leq \zeta \leq 1\}$, $\overline{\Delta} = \{(\zeta, \eta) | 0 \leq \eta \leq \zeta \leq 1\}$ and $\overline{\Delta}_0 = \overline{\Delta} \setminus \{(0,0)\}$.

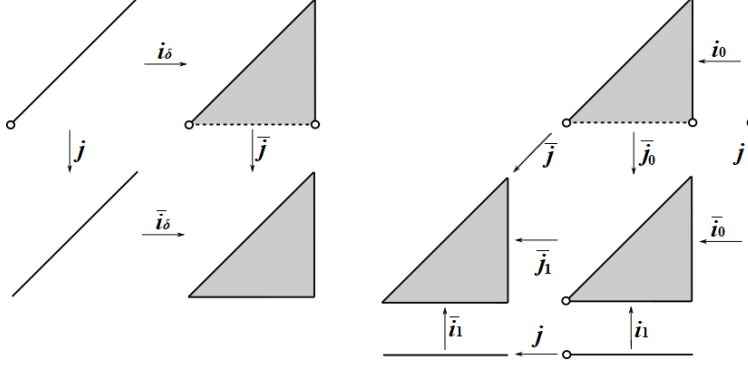


FIGURE 1. The diagram of maps in the proof of Theorem 1.1.

Proof of Theorem 1.1. Consider the 2-parameter family of contact flows $\phi_{\bar{Z}}^{\zeta, \eta}$ ($(\zeta, \eta) \in \Delta$). By Theorem A.2 Remark A.2, for $\mathcal{F} \in \mu Sh_{\mathfrak{c}_{X_0}}(\mathfrak{c}_{X_0})$, we can get a sheaf

$$\Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F}) \in \mu Sh_{(\mathfrak{c}_{X_0})_{\bar{Z}}}((\mathfrak{c}_{X_0})_{\bar{Z}}),$$

where $(\mathfrak{c}_{X_0})_{\bar{Z}'}$ is the Legendrian movie of $\mathfrak{c}_X \cup \Lambda_0 \times \mathbb{R} \cup \tilde{L}_0 \cup \tilde{L}_1$ under the contact flow $\phi_{\bar{Z}}^{\zeta, \eta}$ (in Definition A.1). Applying the antimicrolocalization theorem [9, Theorem 6.28], we write $\Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F})_{\text{dbl}} \in Sh(N \times \Delta)$ for the image of $\Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F})$ under the antimicrolocalization functor.

From Figure 1 one can notice that Φ_{02} and $\Phi_{12} \circ \Phi_{01}$ are (compositions of) nearby cycles along different boundary edges of $\bar{\Delta}$. Therefore it suffices to show that the nearby cycle functors commute and they agree with the 2-parametric nearby cycle functor. In order to apply Lemma 1.2 in our argument, note that firstly $SS^\infty(\Psi_{\bar{Z}'}^{\zeta, \eta}(\mathcal{F})) \cap \pi^* T^{*, \infty} \Delta = \emptyset$ since the singular support is the Legendrian movie under a contact flow, and secondly $SS_\pi^\infty(\Psi_{\bar{Z}'}^{\zeta, \eta}(\mathcal{F})) \cap T^{*, \infty}([0, 1] \times \{0\})$ is subanalytic Legendrian by the fact that

$$\begin{aligned} \lim_{\eta, \zeta \rightarrow 0} \phi_{\bar{Z}'}^{\zeta, \eta}(\mathfrak{c}_{X_0}) &\subseteq \mathfrak{c}_{X_2}, \\ \lim_{\eta \rightarrow 0} \phi_{\bar{Z}'}^{\zeta, \eta}(\mathfrak{c}_{X_0}) &\subseteq \phi_{\bar{Z}}^{\zeta}(\mathfrak{c}_{X_1}), \end{aligned}$$

where the right hand sides are subanalytic Legendrian. Therefore, in all following cases Lemma 1.2 will apply.

(1) Firstly, we consider $\Phi_{02}(\mathcal{F})$ (Figure 1 left). Note that $\varphi_{\bar{Z}_2}^z$ compresses \mathfrak{c}_{X_0} to \mathfrak{c}_{X_2} . Write $i_\delta : N \times (0, 1] \hookrightarrow N \times \Delta$, $(x, \zeta) \mapsto (x, \zeta, \zeta)$, $j : N \times (0, 1] \hookrightarrow N \times [0, 1]$ and $i : N \times \{0\} \hookrightarrow N \times [0, 1]$. Then since $\phi_{\bar{Z}}^{\zeta, \zeta} = \phi_{\bar{Z}}^{\zeta}$,

$$\Phi_{02}(\mathcal{F})_{\text{dbl}} \xrightarrow{\sim} i^{-1} j_* \Psi_{\bar{Z}_2}^{\zeta}(\mathcal{F})_{\text{dbl}} \xrightarrow{\sim} i^{-1} j_* (i_\delta^{-1} \Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F}))_{\text{dbl}}.$$

Write $\bar{i}_\delta : N \times [0, 1] \hookrightarrow N \times \bar{\Delta}$, $(x, \zeta) \mapsto (x, \zeta, \zeta)$, $\bar{j} : N \times \Delta \rightarrow N \times \bar{\Delta}$ and $\bar{i} : N \times \{(0, 0)\} \hookrightarrow N \times \bar{\Delta}$. By Lemma 1.2 and Remark A.3, we know that in fact

$$\Phi_{02}(\mathcal{F})_{\text{dbl}} \xrightarrow{\sim} i^{-1} \bar{i}_\delta^{-1} \bar{j}_* \Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F})_{\text{dbl}} \xrightarrow{\sim} \bar{i}^{-1} \bar{j}_* \Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F})_{\text{dbl}}.$$

(2) Secondly, we consider $\Phi_{12}(\mathcal{F})$ (Figure 1 right). Note that $\varphi_{\bar{Z}_1}^y$ compresses \mathfrak{c}_{X_0} to \mathfrak{c}_{X_1} . Therefore,

$$\Phi_{01}(\mathcal{F})_{\text{dbl}} \xrightarrow{\sim} i^{-1} j_* \Psi_{\bar{Z}_1}^{\eta}(\mathcal{F})_{\text{dbl}}.$$

Write $i_0 : N \times (0, 1] \hookrightarrow N \times \Delta$, $(x, \eta) \mapsto (x, 1, \eta)$. Since $\phi_{\bar{Z}}^{1, \eta} = \phi_{Z'}^{\eta}$, we know that

$$\Phi_{01}(\mathcal{F})_{\text{dbl}} \xrightarrow{\sim} i^{-1} j_* \Psi_{Z_1}^{\eta}(\mathcal{F})_{\text{dbl}} \xrightarrow{\sim} i^{-1} j_* (i_0^{-1} \Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F}))_{\text{dbl}}.$$

Write $\bar{j}_0 : N \times \Delta \hookrightarrow N \times \bar{\Delta}_0$ where $\bar{\Delta}_0 = \bar{\Delta} \setminus \{(0, 0)\}$, and $\bar{i}_0 : N \times [0, 1] \hookrightarrow N \times \bar{\Delta}$, $(x, \eta) \mapsto (x, 1, \eta)$. By Lemma 1.2 and Remark A.3, we know that in fact

$$\Phi_{01}(\mathcal{F})_{\text{dbl}} \xrightarrow{\sim} i^{-1} \bar{i}_0^{-1} \bar{j}_{0,*} \Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F})_{\text{dbl}}.$$

Then we consider $\Phi_{12} \circ \Phi_{01}(\mathcal{F})$ (Figure 1 right). Write $i_1 : N \times (0, 1] \hookrightarrow N \times \bar{\Delta}_0$, $(x, \zeta) \mapsto (x, \zeta, 0)$ where $\bar{\Delta}_0 = \bar{\Delta} \setminus \{(0, 0)\}$. Let $\varphi_{\bar{Z}}^z$ be the contact flow on $T^{*, \infty}(N \times [0, 1])$ defined by the pull back vector field $\pi^* Z_2$ for $\pi : \bar{\Delta}_0 \cong (0, 1] \times [0, 1] \rightarrow (0, 1]$, and $\phi_{\bar{Z}}^{\zeta} = \varphi_{\bar{Z}}^{\ln \zeta}$. Let $\Psi_{\bar{Z}}^{\zeta} : Sh(N \times \{1\} \times [0, 1]) \rightarrow Sh(N \times \bar{\Delta}_0)$ be the Hamiltonian isotopy functor as in Theorem A.1. Thus by Lemma 1.2

$$\begin{aligned} (\Psi_{Z_2}^{\zeta} \circ \Phi_{01}(\mathcal{F}))_{\text{dbl}} &\xrightarrow{\sim} \Psi_{Z_2}^{\zeta} (i^{-1} \bar{i}_0^{-1} \bar{j}_{0,*} \Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F})_{\text{dbl}}) \\ &\xrightarrow{\sim} i_1^{-1} \Psi_{\bar{Z}_{\text{st}}}^{\zeta} (\bar{i}_0^{-1} \bar{j}_{0,*} \Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F})_{\text{dbl}}) \xrightarrow{\sim} i_1^{-1} \bar{j}_{0,*} \Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F})_{\text{dbl}}. \end{aligned}$$

Therefore, by Lemma 1.2 again, we can show that

$$\begin{aligned} \Phi_{12} \circ \Phi_{01}(\mathcal{F})_{\text{dbl}} &\xrightarrow{\sim} i^{-1} j_* (\Psi_{Z_2}^{\zeta} \circ \Phi_{01})(\mathcal{F})_{\text{dbl}} \xrightarrow{\sim} i^{-1} j_* i_1^{-1} \bar{j}_{0,*} \Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F})_{\text{dbl}} \\ &\xrightarrow{\sim} i^{-1} \bar{i}_1^{-1} \bar{j}_{1,*} \bar{j}_{0,*} \Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F})_{\text{dbl}} \xrightarrow{\sim} \bar{i}^{-1} \bar{j}_* \Psi_{\bar{Z}}^{\zeta, \eta}(\mathcal{F})_{\text{dbl}}. \end{aligned}$$

Therefore, we can conclude that $\Phi_{02}(\mathcal{F}) \simeq \Phi_{12} \circ \Phi_{01}(\mathcal{F})$.

(3). On the level of morphisms, the base change formulas provide natural transformations between the morphism spaces, and the gapped full faithfulness theorem for nearby cycles [9, Theorem 4.1] shows that the natural transformations are quasi-isomorphisms, and hence completes the proof. \square

As a corollary, we can immediately get the invariance of the microlocal sheaf category under any Liouville homotopies.

Corollary 1.5. *Let X, X' be Weinstein domains with Lagrangian skeleta $\mathbf{c}_X, \mathbf{c}_{X'}$. Suppose the Liouville forms λ, λ' are homotopic through Liouville forms. Then*

$$\mu Sh_{\mathbf{c}_X}(\mathbf{c}_X) \simeq \mu Sh_{\mathbf{c}_{X'}}(\mathbf{c}_{X'}).$$

Proof. We view X, X' as Weinstein domains with contact boundary. By choosing a sufficiently small Weinstein neighbourhood (with contact boundary) of $\mathbf{c}_{X'}$, we get a Liouville embedding $X' \hookrightarrow X$, and thus a functor

$$\Phi_{X', X} : \mu Sh_{\mathbf{c}_{X'}}(\mathbf{c}_{X'}) \simeq \mu Sh_{\mathbf{c}_X}(\mathbf{c}_X).$$

Then by choosing a sufficiently small Weinstein neighbourhood (with contact boundary) of \mathbf{c}_X , we also get a Liouville embedding $X \hookrightarrow X'$, and thus a functor

$$\Phi_{X, X'} : \mu Sh_{\mathbf{c}_X}(\mathbf{c}_X) \hookrightarrow \mu Sh_{\mathbf{c}_{X'}}(\mathbf{c}_{X'}).$$

Then the theorem implies that $\Phi_{X, X'} \circ \Phi_{X', X} = \text{id}$ and $\Phi_{X', X} \circ \Phi_{X, X'} = \text{id}$. Hence they define inverse equivalences of categories. \square

Remark 1.5. *Oleg Lazarev has pointed out to the author that [5, Proposition 2.42] has shown that for any Liouville homotopy between two different Weinstein structures on X , there is a Weinstein structure on the Liouville movie $X \times T^*[0, 1]$ which agrees with the two Weinstein structures on the two ends. With this proposition, one can show that the argument in [9, Theorem 9.14] implies the above corollary as well. However, to the author's knowledge, when there is only a Liouville embedding of Weinstein manifolds $X_0 \hookrightarrow X_1$, it*

is not true that $X_1 \setminus X_0$ carries a Weinstein structure, and hence for Liouville embeddings, it still seems necessary to use our main result.

APPENDIX A. REVIEW OF HAMILTONIAN INVARIANCE OF SHEAVES

We review the equivalence functors coming from a Hamiltonian isotopy, constructed for sheaves $Sh(M)$ by Guillermou-Kashiwara-Schapira [2], and for microsheaves $\mu Sh_\Lambda(\Lambda)$ by Kashiwara-Schapira [3, Section 7.2]. Throughout the section, we will adapt the definition of microsheaves in [9, Section 5].

Definition A.1. Let $\widehat{H}_s : T^*M \times I \rightarrow \mathbb{R}$ be a homogeneous Hamiltonian on T^*M , and $H_s = \widehat{H}_s|_{T^{*,\infty}M}$ the corresponding contact Hamiltonian on $T^{*,\infty}M$. For a conical Lagrangian $\widehat{\Lambda}$, the Lagrangian movie of $\widehat{\Lambda}$ under the Hamiltonian isotopy $\varphi_{\widehat{H}}^s$ ($s \in I$) is

$$\widehat{\Lambda}_{\widehat{H}} = \{(x, \xi, s, \sigma) | (x, \xi) = \varphi_{\widehat{H}}^s(x_0, \xi_0), \sigma = -\widehat{H}_s \circ \varphi_{\widehat{H}}^s(x_0, \xi_0), (x_0, \xi_0) \in \widehat{\Lambda}\}.$$

For a Legendrian Λ , the Legendrian movie of Λ under the corresponding contact Hamiltonian isotopy is $\Lambda_H = \widehat{\Lambda}_{\widehat{H}} \cap T^{*,\infty}M$.

Theorem A.1 (Guillermou-Kashiwara-Schapira [2, Proposition 3.12]). Let $H_s : T^{*,\infty}M \times I \rightarrow \mathbb{R}$ be a contact Hamiltonian on $T^{*,\infty}M$ and Λ a Legendrian in $T^{*,\infty}M$. Then there are equivalences

$$Sh_\Lambda(M) \xleftarrow{\sim} Sh_{\Lambda_H}(M \times I) \xrightarrow{\sim} Sh_{\varphi_H^1(\Lambda)}(M),$$

given by restriction functors i_0^{-1} and i_1^{-1} where $i_s : M \times \{s\} \hookrightarrow M \times I$ is the inclusion. We denote their inverses by Ψ_H^0 and Ψ_H^1 , and $\Psi_H = i_1^{-1} \circ \Psi_H^0$.

Remark A.1 ([2, Remark 3.9]). This theorem also works for a U -parametric family of Hamiltonian isotopies on $T^{*,\infty}M \times U \rightarrow T^{*,\infty}M$ for a contractible manifold U .

For the category of microlocal sheaves $\mu Sh_\Lambda(\Lambda)$, Kashiwara-Schapira [3, Section 7.2] showed that it is invariant under contact transformations, which are just (local) contactomorphisms. Nadler-Shende explained how this will imply the invariance of $\mu Sh_\Lambda(\Lambda)$ under Hamiltonian isotopies.

Theorem A.2 (Kashiwara-Schapira [3, Theorem 7.2.1], Nadler-Shende [9, Lemma 5.6]). Let $H_s : T^{*,\infty}M \times I \rightarrow \mathbb{R}$ be a contact Hamiltonian on $T^{*,\infty}M$ and Λ a Legendrian in $T^{*,\infty}M$. Then there are equivalences

$$\mu Sh_\Lambda(\Lambda) \xleftarrow{\sim} \mu Sh_{\Lambda_H}(\Lambda_H) \xrightarrow{\sim} \mu Sh_{\varphi_H^1(\Lambda)}(\varphi_H^1(\Lambda))$$

given by restriction functors i_0^{-1} and i_1^{-1} where $i_s : T^{*,\infty}M \times \{s\} \hookrightarrow T^{*,\infty}(M \times I)$ is the inclusion. We denote their inverses by Ψ_H^0 and Ψ_H^1 , and $\Psi_H = i_1^{-1} \circ \Psi_H^0$.

Proof. For any open subset $\Omega \subset T^{*,\infty}M$, consider the contact movie $\Omega_{H,s,\epsilon} \subset T^{*,\infty}(M \times I)$ in the time interval $I_{s,\epsilon} = (s - \epsilon, s + \epsilon)$. Then i_s^{-1} induces equivalences of categories

$$Sh_{\Lambda_H \cup \Omega_{H,s,\epsilon}^c}(M \times I_{s,\epsilon}) \xrightarrow{\sim} Sh_{\varphi_H^s(\Lambda \cup \Omega^c)}(M), \quad Sh_{\Omega_{H,s,\epsilon}^c}(M \times I_{s,\epsilon}) \xrightarrow{\sim} Sh_{\varphi_H^s(\Omega^c)}(M).$$

Since $Sh(M \times I_{s,\epsilon}) = Sh(M \times I) / Sh_{T^*(M \times I \setminus I_{s,\epsilon})}(M \times I)$, we get an equivalence of presheaves

$$i_s^{-1} : \varinjlim_{\epsilon \rightarrow 0} \mu Sh_{\Lambda_H}^{\text{pre}}(\Omega_{H,s,\epsilon}) \xrightarrow{\sim} \mu Sh_{\varphi_H^s(\Lambda)}^{\text{pre}}(\varphi_H^s(\Omega)),$$

where the left hand side is the pull back of a presheaf, since $\Omega_{H,s,\epsilon}$ ($\epsilon > 0$) form a neighbourhood basis of $\varphi_H^s(\Omega)$. Therefore, after sheafification, we can get an equivalence given by the pull back

$$i_s^{-1} : \mu Sh_{\Lambda_H}(\varphi_H^s(\Lambda)) \xrightarrow{\sim} \mu Sh_{\varphi_H^s(\Lambda)}(\varphi_H^s(\Lambda)).$$

Then, since $\mu Sh_{\Lambda_H}^{\text{pre}}(\Omega_{H,s,\epsilon}) \simeq \mu Sh_{\Lambda_H}^{\text{pre}}(\Omega_{H,s',\epsilon})$, we also know that $\mu Sh_{\Lambda_H}^{\text{pre}}$ forms a presheaf that is locally constant in the I direction (along contact movies of points). Since I is contractible, we can conclude that there is an equivalence given by the restriction

$$\mu Sh_{\Lambda_H}(\Lambda_H) \xrightarrow{\sim} \mu Sh_{\varphi_H^s(\Lambda)}(\varphi_H^s(\Lambda)).$$

This completes the proof of the theorem. \square

Remark A.2. *One can show that the theorem also works for a U -parametric family of Hamiltonian isotopies for a contractible manifold U , following Remark A.1. The author would like to thank anonymous referee for pointing out the mistake in the theorem.*

Remark A.3. *From our proof, one may notice that there is a commutative diagram*

$$\begin{array}{ccc} Sh_{\Lambda_H}(M \times I) & \xrightarrow{i_s^{-1}} & Sh_{\varphi_H^s(\Lambda)}(M) \\ \downarrow & & \downarrow \\ \mu Sh_{\Lambda_H}(\Lambda_H) & \xrightarrow{i_s^{-1}} & \mu Sh_{\varphi_H^s(\Lambda)}(\varphi_H^s(\Lambda)). \end{array}$$

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